

Exercise 1.1

Question 1: Determine Whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation R in the set $A = \{1, 2, 3, \dots, 13, 14\}$ defined

$$R = \{(x, y): 3x - y = 0\}$$

(ii) Relation R in the set N of natural numbers defined as

$$R = \{(x, y): y = x + 5 \text{ and } x < 4\}$$

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as

$$R = \{(x, y): y \text{ is divisible by } x\}$$

(iv) Relation R in the set Z of all integers defined as

$$R = \{(x, y): x - y \text{ is an integer}\}$$

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

(b) $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

(c) $R = \{(x, y): x \text{ is exactly 7 cm taller than } y\}$

(d) $R = \{(x, y): x \text{ is wife of } y\}$

(e) $R = \{(x, y): x \text{ is father of } y\}$

Solution 1:

(i) $A = \{1, 2, 3, \dots, 13, 14\}$

$$R = \{(x, y): 3x - y = 0\}$$

$$\therefore R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$$

R is not reflexive since $(1, 1), (2, 2), \dots, (14, 14) \notin R$.

Also, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. [$3(3) - 1 \neq 0$]

Also, R is not transitive as $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$. [$3(1) - 9 \neq 0$]

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) $R = \{(x, y): y = x + 5 \text{ and } x < 4\} = \{(1, 6), (2, 7), (3, 8)\}$

It is clear that $(1, 1) \notin R$.

$\therefore R$ is not reflexive.

$(1, 6) \in R$ But, $(6, 1) \notin R$.

$\therefore R$ is not symmetric.

Now, since there is no pair in R such that (x, y) and $(y, z) \in R$, then (x, z) cannot belong to R .

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(iii) $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) : y \text{ is divisible by } x\}$

We know that any number (x) is divisible by itself.

So, $(x, x) \in R$

$\therefore R$ is reflexive.

Now,

$(2, 4) \in R$ [as 4 is divisible by 2]

But, $(4, 2) \notin R$. [as 2 is not divisible by 4]

$\therefore R$ is not symmetric.

Let $(x, y), (y, z) \in R$. Then, y is divisible by x and z is divisible by y .

$\therefore z$ is divisible by x .

$\Rightarrow (x, z) \in R$

$\therefore R$ is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) $R = \{(x, y) : x - y \text{ is an integer}\}$

Now, for every $x \in \mathbf{Z}$, $(x, x) \in R$ as $x - x = 0$ is an integer.

$\therefore R$ is reflexive.

Now, for every $x, y \in \mathbf{Z}$, if $(x, y) \in R$, then $x - y$ is an integer.

$\Rightarrow -(x - y)$ is also an integer.

$\Rightarrow (y - x)$ is an integer.

$\therefore (y, x) \in R$

$\therefore R$ is symmetric.

Now, Let (x, y) and $(y, z) \in R$, where $x, y, z \in \mathbf{Z}$.

$\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow x - z = (x - y) + (y - z)$ is an integer.

$\therefore (x, z) \in R$

$\therefore R$ is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v) (a) $R = \{(x, y): x \text{ and } y \text{ work at the same place}\}$

$\Rightarrow (x, x) \in R$ [as x and x work at the same place]

$\therefore R$ is reflexive.

If $(x, y) \in R$, then x and y work at the same place.

$\Rightarrow y$ and x work at the same place.

$\Rightarrow (y, x) \in R$.

$\therefore R$ is symmetric.

Now, let $(x, y), (y, z) \in R$

$\Rightarrow x$ and y work at the same place and y and z work at the same place.

$\Rightarrow x$ and z work at the same place.

$\Rightarrow (x, z) \in R$

$\therefore R$ is transitive.

Hence, R is reflexive, symmetric and transitive.

(b) $R = \{(x, y): x \text{ and } y \text{ live in the same locality}\}$

Clearly, $(x, x) \in R$ as x and x is the same human being.

$\therefore R$ is reflexive.

If $(x, y) \in R$, then x and y live in the same locality.

$\Rightarrow y$ and x live in the same locality.

$\Rightarrow (y, x) \in R$

$\therefore R$ is symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow x$ and y live in the same locality and y and z live in the same locality.

$\Rightarrow x$ and z live in the same locality.

$\Rightarrow (x, z) \in R$

$\therefore R$ is transitive.

Hence, R is reflexive, symmetric and transitive.

(c) $R = \{(x, y): x \text{ is exactly } 7 \text{ cm taller than } y\}$

Now, $(x, x) \notin R$

Since human being x cannot be taller than himself.

$\therefore R$ is not reflexive.

Now, let $(x, y) \in R$.

$\Rightarrow x$ is exactly 7 cm taller than y .

Then, y is not taller than x . [Since, y is 7 cm smaller than x]

$\therefore (y, x) \notin R$

Indeed if x is exactly 7 cm taller than y , then y is exactly 7 cm shorter than x .

$\therefore R$ is not symmetric.

Now,

Let $(x, y), (y, z) \in R$.

$\Rightarrow x$ is exactly 7 cm taller than y and y is exactly 7 cm taller than z .

$\Rightarrow x$ is exactly 14 cm taller than z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) $R = \{(x, y): x \text{ is the wife of } y\}$

Now,

$(x, x) \notin R$

Since x cannot be the wife of herself.

$\therefore R$ is not reflexive.

Now, let $(x, y) \in R$

$\Rightarrow x$ is the wife of y .

Clearly y is not the wife of x .

$\therefore (y, x) \notin R$

Indeed, if x is the wife of y , then y is the husband of x .

$\therefore R$ is not transitive.

Let $(x, y), (y, z) \in R$

$\Rightarrow x$ is the wife of y and y is the wife of z .

This case is not possible. Also, this does not imply that x is the wife of z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) $R = \{(x, y): x \text{ is the father of } y\}$

$(x, x) \notin R$

As x cannot be the father of himself.

$\therefore R$ is not reflexive.

Now, let $(x, y) \notin R$.

$\Rightarrow x$ is the father of y .

$\Rightarrow y$ cannot be the father of y .

Indeed, y is the son or the daughter of x .

$\therefore (y, x) \notin R$

$\therefore R$ is not symmetric.

Now, let $(x, y) \in R$ and $(y, z) \notin R$.

$\Rightarrow x$ is the father of y and y is the father of z .

$\Rightarrow x$ is not the father of z .

Indeed, x is the grandfather of z .

$\therefore (x, z) \notin R$

$\therefore R$ is not transitive. Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 2: Show that the relation R in the set \mathbf{R} of real numbers, defined as $R = \{(a, b): a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution 2: (i) $R = \{(a, b): a \leq b^2\}$

It can be observed that $(\frac{1}{2}, \frac{1}{2}) \notin R$,

$$\text{since, } \frac{1}{2} > \left(\frac{1}{2}\right)^2$$

R is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4$ But, 4 is not less than 1^2 .

$$\therefore (4, 1) \notin R$$

\therefore R is not symmetric.

Now,

$$(3, 2), (2, 1.5) \in R \quad [\text{as } 3 < 2^2 = 4 \text{ and } 2 < (1.5)^2 = 2.25]$$

$$\text{But, } 3 > (1.5)^2 = 2.25$$

$$\therefore (3, 1.5) \notin R$$

\therefore R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 3: Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b): b = a + 1\}$ is reflexive, symmetric or transitive.

Solution 3: Let $A = \{1, 2, 3, 4, 5, 6\}$.

A relation R is defined on set A as: $R = \{(a, b): b = a + 1\}$

$$\therefore R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

we can find $(a, a) \notin R$, where $a \in A$.

For instance,

$$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) \notin R$$

\therefore R is not reflexive.

It can be observed that $(1, 2) \in R$, but $(2, 1) \notin R$.

\therefore R is not symmetric.

$$\text{Now, } (1, 2), (2, 3) \in R$$

$$\text{But, } (1, 3) \notin R$$

\therefore R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 4: Show that the relation R in \mathbf{R} defined as $R = \{(a, b): a \leq b\}$, is reflexive and transitive but not symmetric.

Solution 4: $R = \{(a, b): a \leq b\}$

Clearly $(a, a) \in R$ [as $a = a$]

\therefore R is reflexive.

Now, $(2, 4) \in R$ (as $2 < 4$)

But, $(4, 2) \notin R$ as 4 is greater than 2.

\therefore R is not symmetric.

Now, let $(a, b), (b, c) \in R$.

Then, $a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

\therefore R is transitive.

Hence R is reflexive and transitive but not symmetric

Question 5: Check whether the relation R in \mathbf{R} defined as $R = \{(a, b): a \leq b^3\}$ is reflexive, symmetric or transitive.

Solution 5: $R = \{(a, b): a \leq b^3\}$

It is observed that $(\frac{1}{2}, \frac{1}{2}) \notin R$, since, $\frac{1}{2} > \left(\frac{1}{2}\right)^3$

\therefore R is not reflexive.

Now, $(1, 2) \in R$ (as $1 < 2^3 = 8$)

But, $(2, 1) \notin R$ (as $2^3 > 1$)

\therefore R is not symmetric.

We have $(3, \frac{3}{2}), (\frac{3}{2}, \frac{6}{5}) \in R$, since $3 < \left(\frac{3}{2}\right)^2$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$

But $(3, \frac{6}{5}) \notin R$ as $3 > \left(\frac{6}{5}\right)^3$

$\therefore R$ is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 6: Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.

Solution 6: Let $A = \{1, 2, 3\}$.

A relation R on A is defined as $R = \{(1, 2), (2, 1)\}$.

It is clear that $(1, 1), (2, 2), (3, 3) \notin R$.

$\therefore R$ is not reflexive.

Now, as $(1, 2) \in R$ and $(2, 1) \in R$, then R is symmetric.

Now, $(1, 2)$ and $(2, 1) \in R$

However, $(1, 1) \notin R$

$\therefore R$ is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

Question 7: Show that the relation R in the set A of all the books in a library of a college, given by $R = \{(x, y): x \text{ and } y \text{ have same number of pages}\}$ is an equivalence relation.

Solution 7: Set A is the set of all books in the library of a college.

$R = \{(x, y): x \text{ and } y \text{ have the same number of pages}\}$

Now, R is reflexive since $(x, x) \in R$ as x and x has the same number of pages.

Let $(x, y) \in R \Rightarrow x$ and y have the same number of pages.

$\Rightarrow y$ and x have the same number of pages.

$\Rightarrow (y, x) \in R$

$\therefore R$ is symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow x$ and y have the same number of pages and y and z have the same number of pages.

$\Rightarrow x$ and z have the same number of pages.

$\Rightarrow (x, z) \in R$

$\therefore R$ is transitive. Hence, R is an equivalence relation.

Question 8: Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$, is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Solution 8: $A = \{1, 2, 3, 4, 5\}$ and $R = \{(a, b) : |a - b| \text{ is even}\}$

It is clear that for any element $a \in A$, we have $|a - a| = 0$ (which is even).

$\therefore R$ is reflexive.

Let $(a, b) \in R$.

$\Rightarrow |a - b| \text{ is even}$

$\Rightarrow |-(a - b)| = |b - a| \text{ is also even}$

$\Rightarrow (b, a) \in R$

$\therefore R$ is symmetric.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow |a - b| \text{ is even and } |b - c| \text{ is even}$

$\Rightarrow (a - b) \text{ is even and } (b - c) \text{ is even}$

$\Rightarrow (a - c) = (a - b) + (b - c) \text{ is even}$ [Sum of two even integers is even]

$\Rightarrow |a - c| \text{ is even.}$

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set $\{1, 3, 5\}$ are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set $\{2, 4\}$ are related to each other as all the elements of this subset are even.

Also, no element of the subset $\{1, 3, 5\}$ can be related to any element of $\{2, 4\}$ as all elements of $\{1, 3, 5\}$ are odd and all elements of $\{2, 4\}$ are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even. [as $1 - 2, 1 - 4, 3 - 2, 3 - 4, 5 - 2$ and $5 - 4$ all are odd]

Question 9: Show that each of the relation R in the set $A = \{x \in \mathbf{Z}: 0 \leq x \leq 12\}$, given by

(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

(ii) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution 9: Set $A = \{x \in \mathbf{Z}: 0 \leq x \leq 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

For any element, $a \in A$, we have $(a, a) \in R$ as $|a - a| = 0$ is a multiple of 4.

$\therefore R$ is reflexive.

Now, let $(a, b) \in R \Rightarrow |a - b|$ is a multiple of 4.

$$\Rightarrow |-(a - b)| = |b - a| \text{ is a multiple of } 4.$$

$$\Rightarrow (b, a) \in R$$

$\therefore R$ is symmetric.

Now, let $(a, b), (b, c) \in R$.

$$\Rightarrow |a - b| \text{ is a multiple of } 4 \text{ and } |b - c| \text{ is a multiple of } 4.$$

$$\Rightarrow (a - b) \text{ is a multiple of } 4 \text{ and } (b - c) \text{ is a multiple of } 4.$$

$$\Rightarrow (a - c) = (a - b) + (b - c) \text{ is a multiple of } 4.$$

$$\Rightarrow |a - c| \text{ is a multiple of } 4.$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

The set of elements related to 1 is $\{1, 5, 9\}$ as

$$|1 - 1| = 0 \text{ is a multiple of } 4.$$

$$|5 - 1| = 4 \text{ is a multiple of } 4.$$

$$|9 - 1| = 8 \text{ is a multiple of } 4.$$

(ii) $R = \{(a, b) : a = b\}$

For any element $a \in A$, we have $(a, a) \in R$, since $a = a$.

$\therefore R$ is reflexive.

Now, let $(a, b) \in R$.

$$\Rightarrow a = b$$

$$\Rightarrow b = a \Rightarrow (b, a) \in R$$

$\therefore R$ is symmetric.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$$\Rightarrow a = b \text{ and } b = c$$

$$\Rightarrow a = c$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

The elements in R that are related to 1 will be those elements from set A which are equal to 1.

Hence, the set of elements related to 1 is $\{1\}$.

Question 10: Given an example of a relation. Which is

- (i) Symmetric but neither reflexive nor transitive.
- (ii) Transitive but neither reflexive nor symmetric.
- (iii) Reflexive and symmetric but not transitive.
- (iv) Reflexive and transitive but not symmetric.
- (v) Symmetric and transitive but not reflexive

Solution 10: (i) Let $A = \{5, 6, 7\}$.

Define a relation R on A as $R = \{(5, 6), (6, 5)\}$.

Relation R is not reflexive as $(5, 5), (6, 6), (7, 7) \notin R$.

Now, as $(5, 6) \in R$ and also $(6, 5) \in R$, R is symmetric.

$$\Rightarrow (5, 6), (6, 5) \in R, \text{ but } (5, 5) \notin R$$

$\therefore R$ is not transitive.

Hence, relation R is symmetric but not reflexive or transitive.

(ii) Consider a relation R in R defined as:

$$R = \{(a, b) : a < b\}$$

For any $a \in R$, we have $(a, a) \notin R$ since a cannot be strictly less than a itself.

In fact, $a = a$.

$\therefore R$ is not reflexive.

Now, $(1, 2) \in R$ (as $1 < 2$)

But, 2 is not less than 1.

$\therefore (2, 1) \notin R$

$\therefore R$ is not symmetric.

Now, let $(a, b), (b, c) \in R$.

$\Rightarrow a < b$ and $b < c$

$\Rightarrow a < c$

$\Rightarrow (a, c) \in R$

$\therefore R$ is transitive.

Hence, relation R is transitive but not reflexive and symmetric.

(iii) Let $A = \{4, 6, 8\}$.

Define a relation R on A as

$A = \{(4, 4), (6, 6), (8, 8), (4, 6), (6, 4), (6, 8), (8, 6)\}$

Relation R is reflexive since for every $a \in A$, $(a, a) \in R$

i.e., $\{(4, 4), (6, 6), (8, 8)\} \in R$.

Relation R is symmetric since $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in R$.

Relation R is not transitive since $(4, 6), (6, 8) \in R$, but $(4, 8) \notin R$.

Hence, relation R is reflexive and symmetric but not transitive.

(iv) Define a relation R in R as:

$R = \{a, b\}: a^3 \geq b^3\}$

Clearly $(a, a) \in R$ as $a^3 = a^3$

$\therefore R$ is reflexive.

Now, $(2, 1) \in R$ [as $2^3 \geq 1^3$]

But, $(1, 2) \notin R$ [as $1^3 < 2^3$]

$\therefore R$ is not symmetric.

Now, Let $(a, b), (b, c) \in R$.

$$\Rightarrow a^3 \geq b^3 \text{ and } b^3 \geq c^3$$

$$\Rightarrow a^3 \geq c^3$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

Hence, relation R is reflexive and transitive but not symmetric.

(v) Let $A = \{-5, -6\}$.

Define a relation R on A as

$$R = \{(-5, -6), (-6, -5), (-5, -5)\}$$

Relation R is not reflexive as $(-6, -6) \notin R$.

It is seen that $(-5, -6), (-6, -5) \in R$. Also, $(-5, -5) \in R$.

\therefore The relation R is transitive.

Hence, relation R is symmetric and transitive but not reflexive.

Question 11: Show that the relation R in the set A of points in a plane given by $R = \{(P, Q) : \text{Distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$, is an equivalence relation. Further, show that the set of all point related to a point $P \neq (0, 0)$ is the circle passing through P with origin as centre.

Solution 11: $R = \{(P, Q) : \text{Distance of point } P \text{ from the origin is the same as the distance of point } Q \text{ from the origin}\}$

Clearly, $(P, P) \in R$ since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

$\therefore R$ is reflexive.

Now, Let $(P, Q) \in R$.

\Rightarrow The distance of point P from the origin is the same as the distance of point Q from the origin.

\Rightarrow The distance of point Q from the origin is the same as the distance of point P from the origin. $\Rightarrow (Q, P) \in R$

$\therefore R$ is symmetric.

Now, Let $(P, Q), (Q, S) \in R$.

\Rightarrow The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

\Rightarrow The distance of points P and S from the origin is the same.

$\Rightarrow (P, S) \in R$

$\therefore R$ is transitive.

Therefore, R is an equivalence relation.

The set of all points related to $P \neq (0, 0)$ will be those points whose distance from the origin is the same as the distance of point P from the origin.

In other words, if $O (0, 0)$ is the origin and $OP = k$, then the set of all points related to P is at a distance of k from the origin.

Hence, this set of points forms a circle with the centre as the origin and this circle passes through point P.

Question 12: Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Solution 12: $R = \{(T_1, T_2): T_1 \text{ is similar to } T_2\}$

R is reflexive since every triangle is similar to itself.

Further,

If $(T_1, T_2) \in R$, then T_1 is similar to T_2 .

$\Rightarrow T_2$ is similar to T_1 .

$\Rightarrow (T_2, T_1) \in R$

$\therefore R$ is symmetric.

Now,

Let $(T_1, T_2), (T_2, T_3) \in R$.

$\Rightarrow T_1$ is similar to T_2 and T_2 is similar to T_3 .

$\Rightarrow T_1$ is similar to T_3 .

$\Rightarrow (T_1, T_3) \in R$

$\therefore R$ is transitive.

Thus, R is an equivalence relation.

Now,

We can observe that $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} \left(= \frac{1}{2} \right)$

∴ The corresponding sides of triangles T_1 and T_3 are in the same ratio.

Then, triangle T_1 is similar to triangle T_3 .

Hence, T_1 is related to T_3 .

Question 13: Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2): P_1 \text{ and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Solution 13: $R = \{(P_1, P_2): P_1 \text{ and } P_2 \text{ have same the number of sides}\}$

R is reflexive,

Since $(P_1, P_1) \in R$, as the same polygon has the same number of sides with itself.

Let $(P_1, P_2) \in R$.

$\Rightarrow P_1$ and P_2 have the same number of sides.

$\Rightarrow P_2$ and P_1 have the same number of sides.

$\Rightarrow (P_2, P_1) \in R$

∴ R is symmetric.

Now,

Let $(P_1, P_2), (P_2, P_3) \in R$.

$\Rightarrow P_1$ and P_2 have the same number of sides.

Also, P_2 and P_3 have the same number of sides.

$\Rightarrow P_1$ and P_3 have the same number of sides.

$\Rightarrow (P_1, P_3) \in R$

∴ R is transitive.

Hence, R is an equivalence relation.

The elements in A related to the right-angled triangle (T) with sides 3, 4, and 5 are those polygons which have 3 sides (Since T is a polygon with 3 sides).

Hence, the set of all elements in A related to triangle T is the set of all triangles.

Question 14: Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2): L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Solution 14: $R = \{(L_1, L_2): L_1 \text{ is parallel to } L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in R$.

Now, let $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is parallel to $L_2 \Rightarrow L_2$ is parallel to L_1 .

$\Rightarrow (L_2, L_1) \in R$

$\therefore R$ is symmetric.

Now, let $(L_1, L_2), (L_2, L_3) \in R$.

$\Rightarrow L_1$ is parallel to L_2 . Also, L_2 is parallel to L_3 .

$\Rightarrow L_1$ is parallel to L_3 .

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbb{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbb{R}$.

Question 15: Let R be the relation in the set $\{1, 2, 3, 4\}$ given by

$R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation

Solution 15: $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$

It is seen that $(a, a) \in R$, for every $a \in \{1, 2, 3, 4\}$.

$\therefore R$ is reflexive.

It is seen that $(1, 2) \in R$, but $(2, 1) \notin R$.

$\therefore R$ is not symmetric.

Also, it is observed that $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in \{1, 2, 3, 4\}$.

$\therefore R$ is transitive.

Hence, R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16: Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer. (A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$

Solution 16: $R = \{(a, b) : a = b - 2, b > 6\}$

Now,

Since $b > 6$, $(2, 4) \notin R$

Also, as $3 \neq 8 - 2$,

$\therefore (3, 8) \notin R$ And, as $8 \neq 7 - 2$

$\therefore (8, 7) \notin R$

Now, consider $(6, 8)$.

We have $8 > 6$ and also, $6 = 8 - 2$.

$\therefore (6, 8) \in R$

The correct answer is C.

Exercise 1.2

Question 1: Show that the function $f: \mathbf{R}_* \rightarrow \mathbf{R}_*$ defined by $(x) = \frac{1}{x}$ is one-one and onto, where

\mathbf{R}_* is the set of all non-zero real numbers. Is the result true, if the domain \mathbf{R}_* is replaced by N with co-domain being same as \mathbf{R}_* ?

Solution 1: It is given that $f: \mathbb{R}^* \rightarrow \mathbb{R}^*$ is defined by $f(x) = \frac{1}{x}$

For one – one:

Let $x, y \in \mathbb{R}^*$ such that $f(x) = f(y)$

$$\Rightarrow \frac{1}{x} = \frac{1}{y}$$

$$\Rightarrow x = y$$

$\therefore f$ is one – one.

For onto:

It is clear that for $y \in \mathbb{R}^*$, there exists $x = \frac{1}{y} \in \mathbb{R}^*$ [as $y \neq 0$] such that

$$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$$

$\therefore f$ is onto.

Thus, the given function f is one – one and onto.

Now, consider function $g: \mathbb{N} \rightarrow \mathbb{R}^*$ defined by $g(x) = \frac{1}{x}$

$$\text{We have, } g(x_1) = g(x_2) \quad \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \quad \Rightarrow x_1 = x_2$$

$\therefore g$ is one – one.

Further, it is clear that g is not onto as for $1.2 \in \mathbb{R}^*$ there does not exist any x in \mathbb{N} such that $g(x) = \frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Question 2: Check the injectivity and surjectivity of the following functions:

(i) $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^2$

(ii) $f: \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x^2$

(iii) $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$

(iv) $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3$

(v) $f: \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x^3$

Solution 2: (i) $f: \mathbb{N} \rightarrow \mathbb{N}$ is given by $f(x) = x^2$

It is seen that for $x, y \in \mathbb{N}$, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in \mathbb{N}$. But, there does not exist any x in \mathbb{N} such that $f(x) = x^2 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

(ii) $f: \mathbb{Z} \rightarrow \mathbb{Z}$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in \mathbb{Z}$. But, there does not exist any element $x \in \mathbb{Z}$ such that

$f(x) = -2$ or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iii) $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in \mathbb{R}$. But, there does not exist any element $x \in \mathbb{R}$ such that $f(x) = -2$

or $x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iv) $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3$

It is seen that for $x, y \in \mathbb{N}$, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in \mathbb{N}$. But, there does not exist any element $x \in \mathbb{N}$ such that

$$f(x) = 2 \text{ or } x^3 = 2.$$

$\therefore f$ is not surjective

Hence, function f is injective but not surjective.

(v) $f: Z \rightarrow Z$ is given by $f(x) = x^3$

It is seen that for $x, y \in Z$, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in Z$. But, there does not exist any element $x \in Z$ such that

$$f(x) = 2 \text{ or } x^3 = 2.$$

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

Question 3: Prove that the Greatest Integer Function $f: R \rightarrow R$ given by $f(x) = [x]$, is neither one – one nor onto, where $[x]$ denotes the greatest integer less than or equal to x .

Solution 3: $f: R \rightarrow R$ is given by, $f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

$\therefore f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

$\therefore f$ is not one – one.

Now, consider $0.7 \in R$.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in R$ such that $f(x) = 0.7$.

$\therefore f$ is not onto.

Hence, the greatest integer function is neither one – one nor onto.

Question 4: In Show that the Modulus Function $f: R \rightarrow R$ given by $f(x) = |x|$, is neither one – one nor onto, where $|x|$ is x , if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution 4: $f: R \rightarrow R$ is given by $f(x) = |x| = \begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X < 0 \end{cases}$

It is clear that $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$

$\therefore f(-1) = f(1)$, but $-1 \neq 1$.

$\therefore f$ is not one – one.

Now, consider $-1 \in \mathbb{R}$.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element x in domain \mathbb{R} such that $f(x) = |x| = -1$.

$\therefore f$ is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5: Show that the Signum Function $f: \mathbb{R} \rightarrow \mathbb{R}$, given by $f(x) = \begin{cases} 1 & \text{if } X > 0 \\ 0, & \text{if } X = 0 \\ -1, & \text{if } X < 0 \end{cases}$

is neither one-one nor onto.

Solution 5: $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = \begin{cases} 1 & \text{if } X > 0 \\ 0, & \text{if } X = 0 \\ -1, & \text{if } X < 0 \end{cases}$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

$\therefore f$ is not one – one.

Now, as $f(x)$ takes only 3 values (1, 0, or -1) for the element -2 in co-domain

\mathbb{R} , there does not exist any x in domain \mathbb{R} such that $f(x) = -2$.

$\therefore f$ is not onto.

Hence, the Signum function is neither one – one nor onto.

Question 6: Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B . Show that f is one – one.

Solution 6: It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \rightarrow B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

$$\therefore f(1) = 4, f(2) = 5, f(3) = 6$$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one – one.

Question 7: In each of the following cases, state whether the function is one – one, onto or bijective. Justify your answer.

(i) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3 - 4x$

(ii) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1 + x^2$

Solution 7: (i) $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$

$$\Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$$\Rightarrow -4x_1 = -4x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one – one.

For any real number (y) in \mathbb{R} , there exists $\frac{3-y}{4}$ in \mathbb{R} such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$

$\therefore f$ is onto.

Hence, f is bijective.

(ii) $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = 1 + x^2$

Let $x^1, x^2 \in \mathbb{R}$ such that $f(x^1) = f(x^2)$

$$\Rightarrow 1 + X_1^2 = 1 + x_2^2$$

$$\Rightarrow + X_1^2 = X_2^2$$

$$\Rightarrow x_1 = \pm x_2$$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

For example $f(1) = f(-1) = 2$

$\therefore f$ is not one – one.

Consider an element -2 in co-domain \mathbb{R} .

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in \mathbb{R}$.

Thus, there does not exist any x in domain \mathbb{R} such that $f(x) = -2$.

$\therefore f$ is not onto.

Hence, f is neither one – one nor onto.

Question 8: Let A and B be sets. Show that $f: A \times B \rightarrow B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

Solution 8: $f: A \times B \rightarrow B \times A$ is defined as $f(a, b) = (b, a)$.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$$\Rightarrow (b_1, a_1) = (b_2, a_2)$$

$$\Rightarrow b_1 = b_2 \text{ and } a_1 = a_2$$

$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

$\therefore f$ is one – one.

Now, let $(b, a) \in B \times A$ be any element.

Then, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. [By definition of f]

$\therefore f$ is onto.

Hence, f is bijective.

Question 9: Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in \mathbb{N}$

State whether the function f is bijective. Justify your answer.

Solution 9: $f: \mathbb{N} \rightarrow \mathbb{N}$ is defined as $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in \mathbb{N}$

It can be observed that:

$$f(1) = \frac{1+1}{2} = 1 \text{ and } f(2) = \frac{2}{2} = 1 \quad [\text{By definition of } f(n)]$$

$$f(1) = f(2), \text{ where } 1 \neq 2$$

$\therefore f$ is not one-one.

Consider a natural number (n) in co-domain \mathbf{N} .

Case I: n is odd

$\therefore n = 2r + 1$ for some $r \in \mathbf{N}$. Then, there exists $4r + 1 \in \mathbf{N}$ such that

$$f(4r + 1) = \frac{4r + 1 + 1}{2} = 2r + 1$$

Case II: n is even

$\therefore n = 2r$ for some $r \in \mathbf{N}$. Then, there exists $4r \in \mathbf{N}$ such that

$$f(4r) = \frac{4r}{2} = 2r.$$

$\therefore f$ is onto.

Hence, f is not a bijective function.

Question 10: Let $A = \mathbf{R} - \{3\}$ and $B = \mathbf{R} - \{1\}$. Consider the function $f: A \rightarrow B$ defined by $(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution 10: $A = \mathbf{R} - \{3\}$, $B = \mathbf{R} - \{1\}$ and $f: A \rightarrow B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

Let $x, y \in A$ such that $f(x) = f(y)$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow -3x - 2y = -2x - 3y \Rightarrow x = y$$

$\therefore f$ is one-one.

Let $y \in B = \mathbb{R} - \{1\}$. Then, $y \neq 1$.

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now, $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y \Rightarrow x(1 - y) = -3y + 2$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in A \quad [y \neq 1]$$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f \frac{2-3y}{1-y} = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1-y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto.

Hence, function f is one – one and onto.

Question 11: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

- (A) f is one-one onto
- (B) f is many-one onto
- (C) f is one-one but not onto
- (D) f is neither one-one nor onto

Solution 11: $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = x^4$.

Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$.

$$\Rightarrow x^4 = y^4$$

$$\Rightarrow x = \pm y$$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one-one.

Consider an element 2 in co-domain \mathbf{R} . It is clear that there does not exist any x in domain \mathbf{R} such that $f(x) = 2$.

$\therefore f$ is not onto.

Hence, function f is neither one – one nor onto.

The correct answer is D.

Question 12: Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

- (A) f is one – one onto
- (B) f is many – one onto
- (C) f is one – one but not onto
- (D) f is neither one – one nor onto

Solution 12: $f: \mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = 3x$.

Let $x, y \in \mathbf{R}$ such that $f(x) = f(y)$.

$$\Rightarrow 3x = 3y$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

Also, for any real number (y) in co-domain \mathbf{R} , there exists $\frac{y}{3}$ in \mathbf{R} such that $f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$

$\therefore f$ is onto.

Hence, function f is one – one and onto.

The correct answer is A.

Exercise 1.3

Question 1: Let $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down $g \circ f$.

Solution 1: The functions $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ are defined as $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$.

$$g \circ f(1) = g[f(1)] = g(2) = 3 \quad [\text{as } f(1) = 2 \text{ and } g(2) = 3]$$

$$g \circ f(3) = g[f(3)] = g(5) = 1 \quad [\text{as } f(3) = 5 \text{ and } g(5) = 1]$$

$$g \circ f(4) = g[f(4)] = g(1) = 3 \quad [\text{as } f(4) = 1 \text{ and } g(1) = 3]$$

$$\therefore g \circ f = \{(1, 3), (3, 1), (4, 3)\}$$

Question 2: Let f, g and h be functions from \mathbf{R} to \mathbf{R} . Show that

$$(f + g) \circ h = f \circ h + g \circ h$$

$$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

Solution 2: To prove: $(f + g) \circ h = f \circ h + g \circ h$

$$\text{LHS} = [(f + g) \circ h](x)$$

$$= (f + g)[h(x)] = f[h(x)] + g[h(x)]$$

$$= (f \circ h)(x) + (g \circ h)(x)$$

$$= \{(f \circ h)(x) + (g \circ h)(x)\} = \text{RHS}$$

$$\therefore \{(f + g) \circ h\}(x) = \{(f \circ h)(x) + (g \circ h)(x)\} \text{ for all } x \in \mathbf{R}$$

Hence, $(f + g) \circ h = f \circ h + g \circ h$

To Prove: $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$

$$\text{LHS} = [(f \cdot g) \circ h](x)$$

$$= (f \cdot g)[h(x)] = f[h(x)] \cdot g[h(x)]$$

$$= (f \circ h)(x) \cdot (g \circ h)(x)$$

$$= \{(f \circ h) \cdot (g \circ h)\}(x) = \text{RHS}$$

$$\therefore [(f \cdot g) \circ h](x) = \{(f \circ h) \cdot (g \circ h)\}(x) \text{ for all } x \in \mathbf{R}$$

Hence, $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$

Question 3: Find $g \circ f$ and $f \circ g$, if

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Solution 3: (i) $f(x) = |x|$ and $g(x) = |5x-2|$

$$\therefore g \circ f(x) = g(f(x)) = g(|x|) = |5|x|-2|$$

$$f \circ g(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$\therefore g \circ f(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$

$$f \circ g(x) = f(g(x)) = f(x^{\frac{1}{3}})^3 = 8(x^{\frac{1}{3}})^3 = 8x$$

Question 4: If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $f \circ f(x) = x$, for all $x \neq \frac{2}{3}$. What is the inverse of f ?

Solution 4: It is given that $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right) = \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4}$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34}$$

$$\therefore f \circ f(x) = x, \text{ for all } x \neq \frac{2}{3}.$$

$$\Rightarrow f \circ f = I_x$$

Hence, the given function f is invertible and the inverse of f is f itself.

Question 5: State with reason whether following functions have inverse

(i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ with

$$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$$

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with

$$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$$

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with

$$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$$

Solution 5: (i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ defined as $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

From the given definition of f , we can see that f is a many one function as

$$f(1) = f(2) = f(3) = f(4) = 10$$

$\therefore f$ is not one – one.

Hence, function f does not have an inverse.

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ defined as

$$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$$

From the given definition of g , it is seen that g is a many one function as $g(5) = g(7) = 4$.

$\therefore g$ is not one – one.

Hence, function g does not have an inverse.

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ defined as

$$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$$

It is seen that all distinct elements of the set $\{2, 3, 4, 5\}$ have distinct images under h .

\therefore Function h is one – one.

Also, h is onto since for every element y of the set $\{7, 9, 11, 13\}$, there exists an element x in the set $\{2, 3, 4, 5\}$, such that $h(x) = y$.

Thus, h is a one – one and onto function.

Hence, h has an inverse.

Question 6: Show that $f: [-1, 1] \rightarrow \mathbb{R}$, given by $f(x) = X(X+2)$ is one – one. Find the inverse of the function $f: [-1, 1] \rightarrow \text{Range } f$.

(Hint: For $y \in \text{Range } f$, $y = (x) = (X+2)$, for some x in $[-1, 1]$, i.e., $x = 2y(1-y)$)

Solution 6: $f: [-1, 1] \rightarrow \mathbb{R}$ is given as $f(x) = X(X+2)$

For one – one

Let $f(x) = f(y)$

$$\Rightarrow (X+2) = (Y+2)$$

$$\Rightarrow xy + 2x = xy + 2y$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one – one function.

It is clear that $f: [-1, 1] \rightarrow \text{Range } f$ is onto.

$\therefore f: [-1, 1] \rightarrow \text{Range } f$ is one – one and onto and therefore, the inverse of the function $f: [-1, 1] \rightarrow \text{Range } f$ exists.

Let $g: \text{Range } f \rightarrow [-1, 1]$ be the inverse of f .

Let y be an arbitrary element of range f .

Since $f: [-1, 1] \rightarrow \text{Range } f$ is onto, we have

$$y = f(x) \text{ for some } x \in [-1, 1]$$

$$\Rightarrow y = \frac{x}{(x+2)}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y}, y \neq 1$$

Now, let us define $g: \text{Range } f \rightarrow [-1, 1]$ as

$$g(y) = \frac{2y}{1-y}, y \neq 1$$

Now,

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{(x+2)}\right) = 2\left(\frac{\frac{x}{x+2}}{1 - \left(\frac{x}{x+2}\right)}\right) = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

and

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y} + 2} = \frac{2y}{2y + 2 - 2y} = \frac{2y}{2} =$$

$$\therefore g \circ f = x = I_{[-1,1]} \text{ and } f \circ g = y = I_{\text{Range } f}$$

$$\therefore f^{-1} = g$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Question 7: Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f .

Solution 7: $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by, $f(x) = 4x + 3$

For one – one

$$\text{Let } f(x) = f(y)$$

$$\Rightarrow 4x + 3 = 4y + 3$$

$$\Rightarrow 4x = 4y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one – one function

For onto

$$\text{For } y \in \mathbb{R}, \text{ let } y = 4x + 3.$$

$$\Rightarrow x = \frac{y-3}{4} \in \mathbb{R}$$

Therefore, for any $y \in \mathbb{R}$, there exists $x = \frac{y-3}{4} \in \mathbb{R}$, such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y.$$

$\therefore f$ is onto.

Thus, f is one – one and onto and therefore, f^{-1} exists.

Let us define $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = \frac{y-3}{4}$

Now,

$$(g \circ f)(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3)-3}{4} = \frac{4x}{4} = x$$

and

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y-3 + 3 = y$$

$$\therefore g \circ f = f \circ g = I_{\mathbb{R}}$$

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \frac{y-3}{4}$.

Question 8: Consider $f: \mathbb{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where \mathbb{R}_+ is the set of all non-negative real numbers.

Solution 8: $f: \mathbb{R}_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

For one – one

$$\text{Let } f(x) = f(y)$$

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \quad [\text{as } x = y \in \mathbb{R}_+]$$

$\therefore f$ is a one – one function.

For onto

$$\text{For } y \in [4, \infty), \text{ let } y = x^2 + 4$$

$$\Rightarrow x^2 = y - 4 \geq 0 \quad [\text{as } y \geq 4]$$

$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in [4, \infty)$, there exists $x = \sqrt{y-4} \in \mathbb{R}_+$, such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$$

$\therefore f$ is onto.

Thus, f is one – one and onto and therefore, f^{-1} exists.

Let us define $g: [4, \infty) \rightarrow \mathbb{R}^+$ by $g(y) = \sqrt{y-4}$

Now,

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4)-4} = \sqrt{\sqrt{x^2}} = x$$

and

$$(f \circ g)(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$$

$$\therefore g \circ f = f \circ g = I_{\mathbb{R}}$$

Hence, f is invertible and the inverse of f is given by $f^{-1}(y) = g(y) = \sqrt{y-4}$

Question 9: Consider $f: \mathbb{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible

$$\text{with } f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3} \right)$$

Solution 9: $f: \mathbb{R}_+ \rightarrow [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

$$\text{Let } y = 9x^2 + 6x - 5$$

$$\Rightarrow y = (3x + 1)^2 - 1 - 5 = (3x + 1)^2 - 6$$

$$\Rightarrow y + 6 = (3x + 1)^2$$

$$\Rightarrow 3x + 1 = \sqrt{y+6} \quad [\text{as } y \geq -5 \Rightarrow y + 6 > 0]$$

$$\Rightarrow x = \left(\frac{(\sqrt{y+6})-1}{3} \right)$$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

$$\text{Let us define } g: [-5, \infty) \rightarrow \mathbb{R}_+ \text{ as } g(y) = \left(\frac{(\sqrt{y+6})-1}{3} \right)$$

$$\text{Now, } (g \circ f)(x) = g(f(x)) = g(9x^2 + 6x - 5) = g((3x + 1)^2 - 6)$$

$$= \sqrt{(3x+1)^2 - 6 + 6} - 1$$

$$= \frac{3x+1-1}{3} = \frac{3x}{3} = X$$

and

$$\begin{aligned} (f \circ g)(y) &= f(g(y)) = \left(\frac{\sqrt{Y+6}-1}{3} \right) = \left[3 \left(\frac{\sqrt{Y+6}-1}{3} \right) + 1^2 \right] - 6 \\ &= (\sqrt{Y+6})^2 - 6 = Y + 6 - 6 = Y \end{aligned}$$

$$\therefore \text{gof} = X = I_X \text{ and } \text{fog} = Y = I_Y$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \left(\frac{\sqrt{Y+6}-1}{3} \right)$$

Question 10: Let $f: X \rightarrow Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f . Then for all $y \in Y$, $f \circ g_1(y) = I_Y(y) = f \circ g_2(y)$. Use one – one ness of f).

Solution 10: Let $f: X \rightarrow Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2)

Then, for all $y \in Y$, we have

$$f \circ g_1(y) = I_Y(y) = f \circ g_2(y)$$

$$\Rightarrow f(g_1(y)) = f(g_2(y))$$

$$\Rightarrow g_1(y) = g_2(y) \quad [\text{as } f \text{ is invertible} \Rightarrow f \text{ is one – one}]$$

$$\Rightarrow g_1 = g_2 \quad [\text{as } g \text{ is one – one}]$$

Hence, f has a unique inverse.

Question 11: Consider $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Solution 11: Function $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ is given by $f(1) = a$, $f(2) = b$, and $f(3) = c$

If we define $g: \{a, b, c\} \rightarrow \{1, 2, 3\}$ as $g(a) = 1, g(b) = 2, g(c) = 3$.

We have

$$(f \circ g)(a) = f(g(a)) = f(1) = a$$

$$(f \circ g)(b) = f(g(b)) = f(2) = b$$

$$(f \circ g)(c) = f(g(c)) = f(3) = c$$

and

$$(g \circ f)(1) = g(f(1)) = g(a) = 1$$

$$(g \circ f)(2) = g(f(2)) = g(b) = 2$$

$$(g \circ f)(3) = g(f(3)) = g(c) = 3$$

$\therefore g \circ f = I_X$ and $f \circ g = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}: \{a, b, c\} \rightarrow \{1, 2, 3\}$ is given by $f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of g .

If we define $h: \{1, 2, 3\} \rightarrow \{a, b, c\}$ as $h(1) = a, h(2) = b, h(3) = c$

We have

$$(g \circ h)(1) = g(h(1)) = g(a) = 1$$

$$(g \circ h)(2) = g(h(2)) = g(b) = 2$$

$$(g \circ h)(3) = g(h(3)) = g(c) = 3$$

and

$$(h \circ g)(a) = h(g(a)) = h(1) = a$$

$$(h \circ g)(b) = h(g(b)) = h(2) = b$$

$$(h \circ g)(c) = h(g(c)) = h(3) = c$$

$\therefore g \circ h = I_X$ and $h \circ g = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$

Question 12: Let $f: X \rightarrow Y$ be an invertible function. Show that the inverse of f^{-1} is f ,

i.e., $(f^{-1})^{-1} = f$.

Solution 12: Let $f: X \rightarrow Y$ be an invertible function.

Then, there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

Here, $f^{-1} = g$.

Now,

$$g \circ f = I_X \text{ and } f \circ g = I_Y$$

$$\Rightarrow f^{-1} \circ f = I_X \text{ and } f \circ f^{-1} = I_Y$$

Hence, $f^{-1}: Y \rightarrow X$ is invertible and f is the inverse of f^{-1} i.e., $(f^{-1})^{-1} = f$

Question 13: If $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = (3-x^3)^{\frac{1}{3}}$, then $f \circ f(x)$ is

(A) $\frac{1}{x^3}$

(B) x^3

(C) x

(D) $(3-x^3)$

Solution 13: $f: \mathbb{R} \rightarrow \mathbb{R}$ be given as $f(x) = (3-x^3)^{\frac{1}{3}}$

$$\therefore f \circ f(x) = f(f(x)) = f\left((3-x^3)^{\frac{1}{3}}\right) = \left[3 - \left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3 - (3-x^3)\right]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$$

$$\therefore f \circ f(x) = x$$

The correct answer is C.

Question 14: Let $f: \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R}$ be a function as $f(x) = \frac{4x}{3x+4}$. The inverse of f is map g :

Range $f \rightarrow \mathbb{R} - \left\{-\frac{4}{3}\right\}$ given by

$$(A) g(y) = \frac{3y}{3-4y}$$

$$(B) g(y) = \frac{4y}{4-3y}$$

$$(C) g(y) = \frac{4y}{4-3y}$$

$$(D) g(y) = \frac{3y}{4-3y}$$

Solution 14: It is given that $f: \mathbf{R} - \{-\frac{4}{3}\} \rightarrow \mathbf{R}$ be a function as $f(x) = \frac{4x}{3x+4}$

Let y be an arbitrary element of Range f .

Then, there exists $x \in \mathbf{R} - \{-\frac{4}{3}\}$ such that $y = f(x)$

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4-3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4-3y}$$

Let us define $g: \text{Range } f \rightarrow \mathbf{R} - \{-\frac{4}{3}\}$ as $g(y) = \frac{4y}{4-3y}$

Now,

$$\begin{aligned} \text{gof}(x) &= g(f(x)) = g\left(\frac{4x}{3x+4}\right) = \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} \\ &= \frac{16x}{12x+16-12x} = \frac{16x}{16} = x \end{aligned}$$

and

$$\begin{aligned}
 f \circ g(y) &= f(g(y)) = f\left(\frac{4y}{4-3y}\right) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right) + 4} \\
 &= \frac{16y}{12y + 16 - 12y} = \frac{16y}{16} = y
 \end{aligned}$$

$$\therefore \text{gof} = I_{\mathbb{R} - \left\{\frac{4}{3}\right\}} \text{ and } \text{fog} = I_{\text{Range } f}$$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map $g: \text{Range } f \rightarrow \mathbb{R} - \left\{-\frac{4}{3}\right\}$, which is given by $g(y) = \frac{4y}{4-3y}$.

The correct answer is B.

Exercise 1.4

Question 1: Determine whether or not each of the definition of given below gives a binary operation.

In the event that $*$ is not a binary operation, give justification for this.

- (i) On \mathbb{Z}^+ , define $*$ by $a * b = a - b$
- (ii) On \mathbb{Z}^+ , define $*$ by $a * b = ab$
- (iii) On \mathbb{R} , define $*$ by $a * b = ab^2$
- (iv) On \mathbb{Z}^+ , define $*$ by $a * b = |a - b|$
- v) On \mathbb{Z}^+ , define $*$ by $a * b = a$

Solution 1: (i) On \mathbb{Z}^+ , $*$ is defined by $a * b = a - b$.

It is not a binary operation as the image of $(1, 2)$ under $*$ is $1 * 2 = 1 - 2 = -1 \notin \mathbb{Z}^+$.

(ii) On \mathbb{Z}^+ , $*$ is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbb{Z}^+$, there is a unique element ab in \mathbb{Z}^+ .

This means that $*$ carries each pair (a, b) to a unique element $a * b = ab$ in \mathbb{Z}^+ .

Therefore, $*$ is a binary operation.

(iii) On \mathbb{R} , $*$ is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbb{R}$, there is a unique element ab^2 in \mathbb{R} .

This means that $*$ carries each pair (a, b) to a unique element $a * b = ab^2$ in \mathbb{R} .

Therefore, $*$ is a binary operation.

(iv) On \mathbb{Z}^+ , $*$ is defined by $a * b = |a - b|$.

It is seen that for each $a, b \in \mathbb{Z}^+$, there is a unique element $|a - b|$ in \mathbb{Z}^+ .

This means that $*$ carries each pair (a, b) to a unique element $a * b = |a - b|$ in \mathbb{Z}^+ . Therefore, $*$ is a binary operation.

(v) On \mathbb{Z}^+ , $*$ is defined by $a * b = a$.

It is seen that for each $a, b \in \mathbb{Z}^+$, there is a unique element a in \mathbb{Z}^+ .

This means that $*$ carries each pair (a, b) to a unique element $a * b = a$ in \mathbb{Z}^+ .

Therefore, $*$ is a binary operation.

Question 2: For each binary operation $*$ defined below, determine whether $*$ is commutative or associative.

(i) On \mathbb{Z} , define $a * b = a - b$

(ii) On \mathbb{Q} , define $a * b = ab + 1$

(iii) On \mathbb{Q} , define $a * b = ab^2$

(iv) On \mathbb{Z}^+ , define $a * b = 2ab$

(v) On \mathbb{Z}^+ , define $a * b = ab$

(vi) On $\mathbb{R} - \{-1\}$, define $a * b = ab + 1$

Solution 2: (i) On \mathbb{Z} , $*$ is defined by $a * b = a - b$.

It can be observed that $1 * 2 = 1 - 2 = -1$ and $2 * 1 = 2 - 1 = 1$.

$\therefore 1 * 2 \neq 2 * 1$, where $1, 2 \in \mathbb{Z}$

Hence, the operation $*$ is not commutative.

Also, we have

$$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$$

$$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in \mathbb{Z}$

Hence, the operation $*$ is not associative.

(ii) On \mathbb{Q} , $*$ is defined by $a * b = ab + 1$.

It is known that: $ab = ba$ for all $a, b \in \mathbb{Q}$

$$\Rightarrow ab + 1 = ba + 1 \text{ for all } a, b \in \mathbb{Q}$$

$$\Rightarrow a * b = a * b \text{ for all } a, b \in \mathbb{Q}$$

Therefore, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$$

$$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in \mathbb{Q}$

Therefore, the operation $*$ is not associative.

(iii) On \mathbb{Q} , $*$ is defined by $a * b = \frac{ab}{2}$

It is known that: $ab = ba$ for all $a, b \in \mathbb{Q}$

$$\Rightarrow \frac{ab}{2} = \frac{ab}{2} \text{ for all } a, b \in \mathbb{Q}$$

$$\Rightarrow a * b = b * a \text{ for all } a, b \in \mathbb{Q}$$

Therefore, the operation $*$ is commutative.

For all $a, b, c \in \mathbb{Q}$, we have

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

and

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$\therefore (a * b) * c = a * (b * c)$, where $a, b, c \in \mathbb{Q}$

Therefore, the operation $*$ is associative.

(iv) On \mathbb{Z}^+ , $*$ is defined by $a * b = 2ab$.

It is known that: $ab = ba$ for all $a, b \in \mathbb{Z}^+$

$$\Rightarrow 2ab = 2ba \text{ for all } a, b \in \mathbb{Z}^+$$

$$\Rightarrow a * b = b * a \text{ for all } a, b \in \mathbb{Z}^+$$

Therefore, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = 2^{1 \times 2} * 3 = 4 * 3 = 2^{4 \times 3} = 2^{12} \text{ and}$$

$$1 * (2 * 3) = 1 * 22 \times 3 = 1 * 26 = 1 * 64 = 21 \times 64 = 264$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in \mathbb{Z}^+$$

Therefore, the operation $*$ is not associative.

(v) On \mathbb{Z}^+ , $*$ is defined by $a * b = a^b$.

It can be observed that

$$1 * 2 = 1^2 = 1 \text{ and } 2 * 1 = 2^1 = 2$$

$$\therefore 1 * 2 \neq 2 * 1, \text{ where } 1, 2 \in \mathbb{Z}^+$$

Therefore, the operation $*$ is not commutative.

It can also be observed that

$$(2 * 3) * 4 = 2^3 * 4 = 8 * 4 = 8^4 = 2^{12} \text{ and}$$

$$2 * (3 * 4) = 2 * 3^4 = 2 * 81 = 2^{81}$$

$$\therefore (2 * 3) * 4 \neq 2 * (3 * 4), \text{ where } 2, 3, 4 \in \mathbb{Z}^+$$

Therefore, the operation $*$ is not associative.

(vi) On $\mathbb{R} - \{-1\}$ is defined by $a * b = ab + 1$

It can be observed that

$$1 * 2 = 12 + 1 = 13 \text{ and } 2 * 1 = 21 + 1 = 22 = 1$$

$$\therefore 1 * 2 \neq 2 * 1, \text{ where } 1, 2 \in \mathbb{R} - \{-1\}$$

Therefore, the operation $*$ is not commutative.

It can also be observed that

$$(1 * 2) * 3 = \frac{1}{2+1} * 3 = \frac{1}{3} * 3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$\text{and } 1 * (2 * 3) = 1 * \frac{2}{3+1} = 1 * \frac{2}{4} = 1 * \frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$, where $1, 2, 3 \in \mathbb{R} - \{-1\}$

Therefore, the operation $*$ is not associative.

Question 3: Consider the binary operation \wedge on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min \{a, b\}$. Write the operation table of the operation \wedge .

Solution 3: The binary operation \wedge on the set $\{1, 2, 3, 4, 5\}$ is defined as $a \wedge b = \min \{a, b\}$ for all $a, b \in \{1, 2, 3, 4, 5\}$.

Thus, the operation table for the given operation \wedge can be given as:

\wedge	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4: Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by the following multiplication table.

(i) Compute $(2 * 3) * 4$ and $2 * (3 * 4)$

(ii) Is $*$ commutative?

(iii) Compute $(2 * 3) * (4 * 5)$.

(Hint: use the following table)

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Solution 4: (i) $(2 * 3) * 4 = 1 * 4 = 1$

$$2 * (3 * 4) = 2 * 1 = 1$$

(ii) For every $a, b \in \{1, 2, 3, 4, 5\}$, we have $a * b = b * a$. Therefore, the operation $*$ is commutative.

(iii) $(2 * 3) = 1$ and $(4 * 5) = 1$

$$\therefore (2 * 3) * (4 * 5) = 1 * 1 = 1$$

Question 5: Let $'*$ be the binary operation on the set $\{1, 2, 3, 4, 5\}$ defined by $a *' b = \text{H.C.F. of } a \text{ and } b$. Is the operation $'*$ same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Solution 5: The binary operation $'*$ on the set $\{1, 2, 3, 4, 5\}$ is defined as $a *' b = \text{H.C.F. of } a \text{ and } b$.

The operation table for the operation $'*$ can be given as:

$'*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

We observe that the operation tables for the operations $*$ and $'*$ are the same.

Thus, the operation $'*$ is same as the operation $*$.

Question 6: Let $*$ be the binary operation on \mathbb{N} given by $a * b = \text{L.C.M. of } a \text{ and } b$. Find

(i) $5 * 7, 20 * 16$

(ii) Is $*$ commutative?

(iii) Is $*$ associative?

(iv) Find the identity of $*$ in \mathbb{N}

(v) Which elements of \mathbb{N} are invertible for the operation $*$?

Solution 6: The binary operation $*$ on \mathbb{N} is defined as $a * b = \text{L.C.M. of } a \text{ and } b$.

(i) $5 * 7 = \text{L.C.M. of } 5 \text{ and } 7 = 35$

$20 * 16 = \text{L.C.M. of } 20 \text{ and } 16 = 80$

(ii) It is known that

L.C.M of a and b = L.C.M of b and a for all a, b \in N.

$$\therefore a * b = b * a$$

Thus, the operation * is commutative.

(iii) For a, b, c \in N, we have

$$(a * b) * c = (\text{L.C.M of a and b}) * c = \text{LCM of a, b, and c}$$

$$a * (b * c) = a * (\text{LCM of b and c}) = \text{L.C.M of a, b, and c}$$

$$\therefore (a * b) * c = a * (b * c)$$

Thus, the operation * is associative.

(iv) It is known that:

L.C.M. of a and 1 = a = L.C.M. 1 and a for all a \in N

$$\Rightarrow a * 1 = a = 1 * a \text{ for all } a \in \mathbb{N}$$

Thus, 1 is the identity of * in N.

(v) An element a in N is invertible with respect to the operation * if there exists an element b in N, such that $a * b = e = b * a$.

Here, $e = 1$

This means that L.C.M of a and b = 1 = L.C.M of b and a

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of N with respect to the operation *.

Question 7: Is * defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = \text{L.C.M. of } a \text{ and } b$ a binary operation? Justify your answer.

Solution 7: The operation * on the set $A = \{1, 2, 3, 4, 5\}$ is defined as $a * b = \text{L.C.M. of } a \text{ and } b$.

Then, the operation table for the given operation * can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

It can be observed from the obtained table that

$$3 * 2 = 2 * 3 = 6 \notin A,$$

$$5 * 2 = 2 * 5 = 10 \notin A,$$

$$3 * 4 = 4 * 3 = 12 \notin A,$$

$$3 * 5 = 5 * 3 = 15 \notin A,$$

$$4 * 5 = 5 * 4 = 20 \notin A$$

Hence, the given operation $*$ is not a binary operation.

Question 8: Let $*$ be the binary operation on N defined by $a * b = \text{H.C.F. of } a \text{ and } b$. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N ?

Solution 8: The binary operation $*$ on N is defined as: $a * b = \text{H.C.F. of } a \text{ and } b$

It is known that

H.C.F. of a and $b = \text{H.C.F. of } b \text{ and } a$ for all $a, b \in N$.

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

For $a, b, c \in N$, we have

$$(a * b) * c = (\text{H.C.F. of } a \text{ and } b) * c = \text{H.C.F. of } a, b \text{ and } c$$

$$a * (b * c) = a * (\text{H.C.F. of } b \text{ and } c) = \text{H.C.F. of } a, b, \text{ and } c$$

$$\therefore (a * b) * c = a * (b * c)$$

Thus, the operation $*$ is associative.

Now, an element $e \in N$ will be the identity for the operation $*$ if $a * e = a = e * a$ for all $a \in N$.

But this relation is not true for any $a \in N$.

Thus, the operation $*$ does not have any identity in N .

Question 9: Let $*$ be a binary operation on the set Q of rational numbers as follows:

(i) $a * b = a - b$

(ii) $a * b = a^2 + b^2$

(iii) $a * b = a + ab$

$$(iv) a * b = (a - b)^2$$

$$(v) a * b = \frac{ab}{4}$$

$$(vi) a * b = ab^2$$

Find which of the binary operations are commutative and which are associative.

Solution 9: On \mathbb{Q} , the operation $*$ is defined as $a * b = a - b$. It can be observed that:

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{3} = \frac{1}{3}$$

And

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}, \text{ where } \frac{1}{2}, \frac{1}{3} \in \mathbb{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that

$$\left(\frac{1}{3} * \frac{1}{2}\right) * \frac{1}{4} = \left(\frac{1}{2} - \frac{1}{3}\right) * \frac{1}{4} = \frac{1}{6} * \frac{1}{4} = \frac{1}{6} - \frac{1}{4} = \frac{2-3}{12} = \frac{-1}{12}$$

and

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) = \frac{1}{2} * \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} * \left(\frac{4-3}{12}\right) = \frac{1}{2} * \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{6-1}{12} = \frac{5}{12}$$

$$\therefore \left(\frac{1}{3} * \frac{1}{2}\right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right), \text{ where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in \mathbb{Q}$$

Thus, the operation $*$ is not associative.

(ii) On \mathbb{Q} , the operation $*$ is defined as $a * b = a^2 + b^2$.

For $a, b \in \mathbb{Q}$, we have

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (12 + 22) * 3 = (1 + 4) * 3 = 5 * 3 = 52 + 32 = 34$$

and

$$1 * (2 * 3) = 1 * (22 + 32) = 1 * (4 + 9) = 1 * 13 = 12 + 132 = 170$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in \mathbb{Q}$$

Thus, the operation $*$ is not associative.

(iii) On \mathbb{Q} , the operation $*$ is defined as $a * b = a + ab$.

It can be observed that

$$1 * 2 = 1 + 1 \times 2 = 1 + 2 = 3$$

$$2 * 1 = 2 + 2 \times 1 = 2 + 2 = 4$$

$$\therefore 1 * 2 \neq 2 * 1, \text{ where } 1, 2 \in \mathbb{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that

$$(1 * 2) * 3 = (1 + 1 \times 2) * 3 = (1 + 2) * 3 = 3 * 3 = 3 + 3 \times 3 = 3 + 9 = 12 \text{ and}$$

$$1 * (2 * 3) = 1 * (2 + 2 \times 3) = 1 * (2 + 6) = 1 * 8 = 1 + 1 \times 8 = 1 + 8 = 9$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in \mathbb{Q}$$

Thus, the operation $*$ is not associative.

(iv) On \mathbb{Q} , the operation $*$ is defined by $a * b = (a - b)^2$.

For $a, b \in \mathbb{Q}$, we have

$$a * b = (a - b)^2$$

$$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (1 - 2)^2 * 3 = (-1)^2 * 3 = 1 * 3 = (1 - 3)^2 = (-2)^2 = 4$$

and

$$1 * (2 * 3) = 1 * (2 - 3)^2 = 1 * (-1)^2 = 1 * 1 = (1 - 1)^2 = 0$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in \mathbb{Q}$$

Thus, the operation $*$ is not associative.

(v) On \mathbb{Q} , the operation $*$ is defined as $a * b = \frac{ab}{4}$.

For $a, b \in \mathbb{Q}$, we have

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

For $a, b, c \in \mathbb{Q}$, we have

$$(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{abc}{16}$$

and

$$a * (b * c) = a * \left(\frac{bc}{4}\right) = \frac{a \cdot \frac{bc}{4}}{4} = \frac{abc}{16}$$

$$\therefore (a * b) * c = a * (b * c), \text{ where } a, b, c \in \mathbb{Q}$$

Thus, the operation $*$ is associative.

(vi) On \mathbb{Q} , the operation $*$ is defined as $a * b = ab^2$

It can be observed that

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} * \left(\frac{1}{3}\right)^2 = \frac{1}{2} * \frac{1}{9} = \frac{1}{18}$$

$$\text{And } \frac{1}{3} * \frac{1}{2} = \frac{1}{3} * \left(\frac{1}{2}\right)^2 = \frac{1}{3} * \frac{1}{4} = \frac{1}{12}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}, \text{ where } \frac{1}{2} \text{ and } \frac{1}{3} \in \mathbb{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that

$$\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} = \left[\frac{1}{2} \left(\frac{1}{3}\right)^2\right] * \frac{1}{4} = \frac{1}{18} * \frac{1}{4} = \frac{1}{18} \left(\frac{1}{4}\right)^2 = \frac{1}{18 \times 16} = \frac{1}{288}$$

$$\text{And } \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4} \right) = \frac{1}{2} * \left[\frac{1}{3} \left(\frac{1}{4} \right)^2 \right] = \frac{1}{2} * \frac{1}{48} = \frac{1}{2} \left(\frac{1}{48} \right)^2 = \frac{1}{2 \times 2304} = \frac{1}{4608}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3} \right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4} \right), \text{ where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10: Find which of the operations given above has identity.

Solution 10: An element $e \in \mathbf{Q}$ will be the identity element for the operation $*$

if $a * e = a = e * a$, for all $a \in \mathbf{Q}$.

However, there is no such element $e \in \mathbf{Q}$ with respect to each of the six operations satisfying the above condition.

Thus, none of the six operations has identity.

Question 11: $A = \mathbf{N} \times \mathbf{N}$ and $*$ be the binary operation on A defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Show that $*$ is commutative and associative. Find the identity element for $*$ on A , if any.

Solution 11: $A = \mathbf{N} \times \mathbf{N}$ and $*$ is a binary operation on A and is defined by

$$(a, b) * (c, d) = (a + c, b + d)$$

Let $(a, b), (c, d) \in A$

Then, $a, b, c, d \in \mathbf{N}$

We have:

$$(a, b) * (c, d) = (a + c, b + d)$$

$$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$$

[Addition is commutative in the set of natural numbers]

$$\therefore (a, b) * (c, d) = (c, d) * (a, b)$$

Therefore, the operation $*$ is commutative.

Now, let $(a, b), (c, d), (e, f) \in A$

Then, $a, b, c, d, e, f \in \mathbb{N}$

We have

$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) = (a + c + e, b + d + f)$$

and

$$(a, b) * [(c, d) * (e, f)] = (a, b) * (c + e, d + f) = (a + c + e, b + d + f)$$

$$\therefore [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$$

Therefore, the operation $*$ is associative.

Let an element $e = (e_1, e_2) \in A$ will be an identity element for the operation $*$ if $a * e = a = e * a$ for all $a = (a_1, a_2) \in A$

$$\text{i.e., } (a_1 + e_1, a_2 + e_2) = (a_1, a_2) = (e_1 + a_1, e_2 + a_2)$$

Which is not true for any element in A .

Therefore, the operation $*$ does not have any identity element.

Question 12: State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation $*$ on a set \mathbb{N} , $a * a = a \forall a \in \mathbb{N}$.

(ii) If $*$ is a commutative binary operation on \mathbb{N} , then $a * (b * c) = (c * b) * a$

Solution 12: (i) Define an operation $*$ on \mathbb{N} as $a * b = a + b \forall a, b \in \mathbb{N}$

Then, in particular, for $b = a = 3$, we have

$$3 * 3 = 3 + 3 = 6 \neq 3$$

Therefore, statement (i) is false.

(ii) R.H.S. = $(c * b) * a$

$$= (b * c) * a \quad [* \text{ is commutative}]$$

$$= a * (b * c) \quad [\text{Again, as } * \text{ is commutative}]$$

$$= \text{L.H.S.}$$

$$\therefore a * (b * c) = (c * b) * a$$

Therefore, statement (ii) is true.

Question 13: Consider a binary operation $*$ on \mathbb{N} defined as $a * b = a^3 + b^3$. Choose the correct answer.

- (A) Is $*$ both associative and commutative?
- (B) Is $*$ commutative but not associative?
- (C) Is $*$ associative but not commutative?
- (D) Is $*$ neither commutative nor associative?

Solution 13: On \mathbb{N} , the operation $*$ is defined as $a * b = a^3 + b^3$.

For, $a, b, \in \mathbb{N}$, we have

$$a * b = a^3 + b^3 = b^3 + a^3 = b * a \quad [\text{Addition is commutative in } \mathbb{N}]$$

Therefore, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (1^3 + 2^3) * 3 = (1 + 8) * 3 = 9 * 3 = 9^3 + 3^3 = 729 + 27 = 756 \text{ and}$$

$$1 * (2 * 3) = 1 * (2^3 + 3^3) = 1 * (8 + 27) = 1 * 35 = 1^3 + 35^3 = 1 + 42875 = 42876$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3), \text{ where } 1, 2, 3 \in \mathbb{N}$$

Therefore, the operation $*$ is not associative.

Hence, the operation $*$ is commutative, but not associative.

Thus, the correct answer is B.

Miscellaneous Exercise

Question 1: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 10x + 7$. Find the function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $g \circ f = f \circ g = I_{\mathbb{R}}$.

Solution 1: It is given that $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = 10x + 7$.

For one – one

$$\text{Let } f(x) = f(y), \text{ where } x, y \in \mathbb{R}.$$

$$\Rightarrow 10x + 7 = 10y + 7$$

$$\Rightarrow x = y$$

$\therefore f$ is a one – one function.

For onto

For $y \in \mathbf{R}$, let $y = 10x + 7$.

$$\Rightarrow x = \frac{y-7}{10} \in \mathbf{R}$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-7}{10} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore f$ is onto.

Therefore, f is one – one and onto.

Thus, f is an invertible function.

Let us define $g: \mathbf{R} \rightarrow \mathbf{R}$ as $g(y) = \frac{y-7}{10}$

Now, we have

$$g \circ f(x) = g(f(x)) = g(10x + 7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = x$$

and

$$f \circ g(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore g \circ f = I_{\mathbf{R}}$ and $f \circ g = I_{\mathbf{R}}$.

Hence, the required function $g: \mathbf{R} \rightarrow \mathbf{R}$ is defined as $g(y) = \frac{y-7}{10}$.

Question 2: Let $f: \mathbf{W} \rightarrow \mathbf{W}$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f . Here, \mathbf{W} is the set of all whole numbers.

Solution 2: It is given that:

$$f: \mathbf{W} \rightarrow \mathbf{W} \text{ is defined as } f(n) = \begin{cases} n-1 & \text{If } n \text{ is odd} \\ n+1 & \text{If } n \text{ is even} \end{cases}$$

For one – one

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$$\Rightarrow n - m = 2$$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

\therefore Both n and m must be either odd or even. Now, if both n and m are odd,

Then, we have

$$f(n) = f(m)$$

$$\Rightarrow n - 1 = m - 1$$

$$\Rightarrow n = m$$

Again, if both n and m are even,

Then, we have

$$f(n) = f(m)$$

$$\Rightarrow n + 1 = m + 1$$

$$\Rightarrow n = m$$

$\therefore f$ is one – one.

For onto

It is clear that any odd number $2r + 1$ in co-domain N is the image of $2r$ in domain N and any even number $2r$ in co-domain N is the image of $2r + 1$ in domain N .

$\therefore f$ is onto.

Hence, f is an invertible function.

Let us define $g: W \rightarrow W$ as $(m) = \begin{cases} m+1 & \text{If } m \text{ is even} \\ m-1 & \text{If } m \text{ is odd} \end{cases}$

Now, when n is odd

$$go(n) = g(f(n)) = g(n - 1) = n - 1 + 1 = n \text{ and}$$

When n is even

$$go(n) = g(f(n)) = g(n + 1) = n + 1 - 1 = n$$

Similarly,

When m is odd

$$fo(m) = f(g(m)) = f(m - 1) = m - 1 + 1 = m \text{ and}$$

When m is even

$$fo(m) = f(g(m)) = f(m + 1) = m + 1 - 1 = m$$

$\therefore gof = I_w$ and $fog = I_w$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f .

Hence, the inverse of f is f itself.

Question 3: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution 3: It is given that $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} (f(x)) &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Question 4: Show that function $f: \mathbb{R} \rightarrow \{x \in \mathbb{R}: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$ is one – one and onto function.

Solution 4: It is given that $f: \mathbb{R} \rightarrow \{x \in \mathbb{R}: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$.

For one – one

Suppose $f(x) = f(y)$, where $x, y \in \mathbb{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative,

$$\text{Then, we have } \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative

$$x > y \Rightarrow x - y > 0$$

But, $2xy$ is negative.

Then $2xy \neq x - y$

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out.

$\therefore x$ and y have to be either positive or negative.

When x and y are both positive, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both negative, we have

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$$

$\therefore f$ is one – one.

For onto

Now, let $y \in \mathbb{R}$ such that $-1 < y < 1$.

If y is negative, then, there exists $x = y1+y \in \mathbb{R}$ such that

$$(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then, there exists $x = y1-y \in \mathbb{R}$ such that

$$(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{-y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto.

Hence, f is one – one and onto.

Question 5: Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3$ is injective.

Solution 5: $f: \mathbb{R} \rightarrow \mathbb{R}$ is given as $f(x) = x^3$.

For one – one

Suppose $f(x) = f(y)$, where $x, y \in \mathbb{R}$.

$$\Rightarrow x^3 = y^3 \quad (1)$$

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$$\Rightarrow x^3 \neq y^3$$

However, this will be a contradiction to (1).

$\therefore x = y$ Hence, f is injective.

Question 6: Give examples of two functions $f: \mathbb{N} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g \circ f$ is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Solution 6: Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as $f(x) = x$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that

$$g(-1) = |-1| = 1$$

$$g(1) = |1| = 1$$

$$\therefore g(-1) = g(1), \text{ but } -1 \neq 1.$$

$\therefore g$ is not injective.

Now, $g \circ f$ of: $\mathbb{N} \rightarrow \mathbb{Z}$ is defined as $g \circ f(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in \mathbb{N}$ such that $g \circ f(x) = g \circ f(y)$.

$$\Rightarrow |x| = |y|$$

Since x and $y \in \mathbb{N}$, both are positive.

$$\therefore |x| = |y| \Rightarrow x = y$$

Hence, $g \circ f$ is injective.

Question 7: Given examples of two functions $f: \mathbb{N} \rightarrow \mathbb{N}$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ such that $g \circ f$ is onto but f is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1, & \text{If } x > 1 \\ 1, & \text{If } x = 1 \end{cases}$)

Solution 7: Define $f: \mathbb{N} \rightarrow \mathbb{N}$ by $f(x) = x + 1$

and $g: \mathbb{N} \rightarrow \mathbb{N}$ by $g(x) = \begin{cases} x-1, & \text{If } x > 1 \\ 1, & \text{If } x = 1 \end{cases}$

We first show that g is not onto.

For this, consider element 1 in co-domain \mathbb{N} . It is clear that this element is not an image of any of the elements in domain \mathbb{N} .

$\therefore g$ is not onto.

Now, $g \circ f: \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$g \circ f(x) = g(f(x)) = g(x + 1) = x + 1 - 1 = x \quad [x \in \mathbb{N} \Rightarrow x + 1 > 1]$$

Then, it is clear that for $y \in \mathbb{N}$, there exists $x = y \in \mathbb{N}$ such that $g \circ f(x) = y$.

Hence, $g \circ f$ is onto.

Question 8: Given a non-empty set X , consider $P(X)$ which is the set of all subsets of X . Define the relation R in $P(X)$ as follows:
For subsets A, B in $P(X)$, $A R B$ if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify your answer.

Solution 8: Since every set is a subset of itself, $A R A$ for all $A \in P(X)$.

$\therefore R$ is reflexive.

Let $A R B \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A .

$\therefore R$ is not symmetric.

Further, if $A R B$ and $B R C$, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow A R C$

$\therefore R$ is transitive.

Hence, R is not an equivalence relation as it is not symmetric.

Question 9: Given a non-empty set X , consider the binary operation $*$: $P(X) \times P(X) \rightarrow P(X)$ given by

$A * B = A \cap B \forall A, B$ in $P(X)$ is the power set of X . Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

Solution 9: It is given the binary operation $*$:

$P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \forall A, B$ in $P(X)$

We know that $A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$

Thus, X is the identity element for the given binary operation $*$.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$ [As X is the identity element]

or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in $P(X)$ with respect to the given operation $*$.

Hence, the given result is proved.

Question 10: Find the number of all onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself.

Solution 10: Onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself is simply a permutation on n symbols

$1, 2, \dots, n$.

Thus, the total number of onto maps from $\{1, 2, \dots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, \dots, n$, which is $n!$.

Question 11: Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T , if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$

(ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

Solution 11: $S = \{a, b, c\}$, $T = \{1, 2, 3\}$

(i) $F: S \rightarrow T$ is defined as $F = \{(a, 3), (b, 2), (c, 1)\}$

$\Rightarrow F(a) = 3, F(b) = 2, F(c) = 1$

Therefore, $F^{-1}: T \rightarrow S$ is given by $F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) $F: S \rightarrow T$ is defined as $F = \{(a, 2), (b, 1), (c, 1)\}$

Since $F(b) = F(c) = 1$, F is not one – one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12: Consider the binary operations $*$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and \circ : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined as $a * b = |a - b|$ and $a \circ b = a$, $\forall a, b \in \mathbb{R}$. Show that $*$ is commutative but not associative, \circ is associative but not commutative. Further, show that $\forall a, b, c \in \mathbb{R}$, $a*(b \circ c) = (a * b) \circ (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation \circ]. Does \circ distribute over $*$? Justify your answer.

Solution 12: It is given that $*$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and \circ : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined as $a * b = |a - b|$ and $a \circ b = a$, $\forall a, b \in \mathbb{R}$

For $a, b \in \mathbb{R}$, we have $a * b = |a - b|$ and $b * a = |b - a| = |-(a - b)| = |a - b| \therefore a * b = b * a$

Hence, the operation $*$ is commutative.

It can be observed that

$$(1 * 2) * 3 = (|1 - 2|) * 3 = 1 * 3 = |1 - 3| = 2$$

and

$$1 * (2 * 3) = 1 * (|2 - 3|) = 1 * 1 = |1 - 1| = 0$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3) \text{ where } 1, 2, 3 \in \mathbb{R}.$$

Hence, the operation $*$ is not associative.

Now, consider the operation \circ

It can be observed that $1 \circ 2 = 1$ and $2 \circ 1 = 2$.

$$\therefore 1 \circ 2 \neq 2 \circ 1 \text{ where } 1, 2 \in \mathbb{R}.$$

Hence, the operation \circ is not commutative.

Let $a, b, c \in \mathbb{R}$. Then, we have

$$(a \circ b) \circ c = a \circ c = a$$

and

$$a \circ (b \circ c) = a \circ b = a$$

$$\therefore (a \circ b) \circ c = a \circ (b \circ c), \text{ where } a, b, c \in \mathbb{R}$$

Hence, the operation \circ is associative.

Now, let $a, b, c \in \mathbb{R}$, then we have

$$a * (b \circ c) = a * b = |a - b|$$

$$(a * b) \circ (a * c) = (|a - b|) \circ (|a - c|) = |a - b|$$

Hence, $a * (b \circ c) = (a * b) \circ (a * c)$.

Now,

$$1 \circ (2 * 3) = 1 \circ (|2 - 3|) = 1 \circ 1 = 1$$

$$(1 \circ 2) * (1 \circ 3) = 1 * 1 = |1 - 1| = 0$$

$\therefore 1 \circ (2 * 3) \neq (1 \circ 2) * (1 \circ 3)$ where $1, 2, 3 \in \mathbb{R}$

Hence, the operation \circ does not distribute over $*$

Question 13: Given a non - empty set X , let $*$: $P(X) \times P(X) \rightarrow P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$. Show that the empty set Φ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

(Hint: $(A - \Phi) \cup (\Phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \Phi$).

Solution 13: It is given that $*$: $P(X) \times P(X) \rightarrow P(X)$ is defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$.

Let $A \in P(X)$. Then, we have

$$A * \Phi = (A - \Phi) \cup (\Phi - A) = A \cup \Phi = A$$

$$\Phi * A = (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$$

$\therefore A * \Phi = A = \Phi * A$ for all $A \in P(X)$

Thus, Φ is the identity element for the given operation $*$.

Now, an element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that $A * B = \Phi = B * A$. [As Φ is the identity element]

Now, we observed that

$$A * A = (A - A) \cup (A - A) = \Phi \cup \Phi = \Phi \text{ for all } A \in P(X).$$

Hence, all the elements A of $P(X)$ are invertible with $A^{-1} = A$.

Question 14: Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \begin{cases} a + b, & \text{If } a + b < 6 \\ a + b - 6, & \text{If } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a .

Solution 14: Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation $*$ on X is defined as $a * b = \begin{cases} a + b, & \text{If } a + b < 6 \\ a + b - 6, & \text{If } a + b \geq 6 \end{cases}$

An element $e \in X$ is the identity element for the operation $*$, if

$$a * e = a = e * a \text{ for all } a \in X$$

For $a \in X$, we have

$$a * 0 = a + 0 = a \quad [a \in X \Rightarrow a + 0 < 6]$$

$$0 * a = 0 + a = a \quad [a \in X \Rightarrow 0 + a < 6]$$

$\therefore a * 0 = a = 0 * a$ for all $a \in X$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

$$\text{i.e., } \begin{cases} a + b = 0 = b + a, & \text{If } a + b < 6 \\ a + b - 6 = 0 = b + a - 6, & \text{If } a + b \geq 6 \end{cases}$$

$$\Rightarrow a = -b \text{ or } b = 6 - a$$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a for all $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15: Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \rightarrow B$ be functions defined by

$$f(x) = x^2 - x, x \in A \text{ and } g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A. \text{ Are } f \text{ and } g \text{ equal? Justify your answer. (Hint:}$$

One may note that two function $f: A \rightarrow B$ and $g: A \rightarrow B$ such that $f(a) = g(a) \forall a \in A$, are called equal functions)

Solution 15: It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \rightarrow B$ are defined by

$$f(x) = x^2 - x, x \in A \text{ and } g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A$$

It is observed that

$$(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$\text{and } g(-1) = 2 \left| (-1) - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow (-1) = g(-1)$$

$$(0) = (0)^2 - (0) = 0$$

$$\text{and } g(0) = 2 \left| 0 - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow (0) = g(0)$$

$$(1) = (1)^2 - (1) = 1 - 1 = 0$$

$$\text{and } g(1) = 2 \left| (1) - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow (1) = g(1)$$

$$(2) = (2)^2 - (2) = 4 - 2 = 2$$

$$\text{And } g(2) = 2 \left| (2) - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow (2) = g(2)$$

$$\therefore (a) = g(a) \text{ for all } a \in A$$

Hence, the functions f and g are equal.

Question 16: Let $A = \{1, 2, 3\}$. Then number of relations containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is

- (A) 1
 (B) 2
 (C) 3
 (D) 4

Solution 16: The given set is $A = \{1, 2, 3\}$.

The smallest relation containing $(1, 2)$ and $(1, 3)$ which is reflexive and symmetric, but not transitive is given by:

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$$

This is because relation R is reflexive as $(1, 1), (2, 2), (3, 3) \in R$.

Relation R is symmetric since $(1, 2), (2, 1) \in R$ and $(1, 3), (3, 1) \in R$.

But relation R is not transitive as $(3, 1), (1, 2) \in R$, but $(3, 2) \notin R$.

Now, if we add any two pairs $(3, 2)$ and $(2, 3)$ (or both) to relation R , then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17: Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing $(1, 2)$ is

- (A) 1
 (B) 2
 (C) 3
 (D) 4

Solution 17: It is given that $A = \{1, 2, 3\}$.

The smallest equivalence relation containing $(1, 2)$ is given by,

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Now, we are left with only four pairs i.e., $(2, 3), (3, 2), (1, 3)$, and $(3, 1)$.

If we add any one pair [say $(2, 3)$] to R_1 , then for symmetry we must add $(3, 2)$.

Also, for transitivity we are required to add $(1, 3)$ and $(3, 1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1, 2)$ is two.

The correct answer is B.

Question 18: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the Signum Function defined as $(x) = \begin{cases} 1, & X > 0 \\ 0, & X = 0 \\ -1, & X < 0 \end{cases}$

and $g: \mathbb{R} \rightarrow \mathbb{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x . Then does $f \circ g$ and $g \circ f$ coincide in $(0, 1]$?

Solution 18: It is given that,

$$f: \mathbb{R} \rightarrow \mathbb{R} \text{ is defined as } f(x) = \begin{cases} 1, & X > 0 \\ 0, & X = 0 \\ -1, & X < 0 \end{cases}$$

Also, $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x .
Now, let $x \in (0, 1]$.

Then, we have

$$[x] = 1 \text{ if } x = 1 \text{ and } [x] = 0 \text{ if } 0 < x < 1.$$

$$f \circ g(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & \text{If } x = 1 \\ f(0), & \text{If } x \in (0, 1) \end{cases}$$

$$= \begin{cases} 1, & \text{If } x = 1 \\ 0, & \text{If } x \in (0, 1) \end{cases}$$

$$g \circ f(x) = g(f(x)) = g(1) \quad [\text{as } x > 0]$$

$$= [1] = 1$$

Thus, when $x \in (0, 1)$, we have $f \circ g(x) = 0$ and $g \circ f(x) = 1$.

Hence, $f \circ g$ and $g \circ f$ do not coincide in $(0, 1]$.

Question 19: Number of binary operations on the set $\{a, b\}$ are

- (A) 10
- (B) 16
- (C) 20
- (D) 8

Solution 19: A binary operation $*$ on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \rightarrow \{a, b\}$
i.e., $*$ is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \rightarrow \{a, b\}$.

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B.
